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THE CALCULATION OF  $e^{At}$  WITH SOME APPLICATIONS

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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>Given A an $n \times n$ matrix with real or complex elements, then $e^{At}$ is calculated as the unique solution of an initial value problem. In the process of obtaining this solution $n$ -unknown matrices become involved and must be computed. Characterizing properties of these matrices to be computed become known: such properties as pairwise-orthogonal, idempotent and nilpotent. Finally some applications of the above calculations are given in the field of solutions to systems of differential equations. |  |                                                                                                           |                               |



# THE CALCULATION OF $e^{At}$ WITH SOME APPLICATIONS

## 1. Introduction

Throughout this paper  $A$  will be an  $n \times n$  matrix with real or complex elements, having  $f(\lambda)$  as its characteristic function and eigenvalues  $\alpha_1, \alpha_2, \dots, \alpha_n$  (not necessarily distinct). In this paper we calculate the exponential matrix  $e^{At}$ , specify some properties of certain matrices that must be determined in order to describe  $e^{At}$  and finally indicate some applications of this calculation.

## 2. An Initial Value Problem and $e^{At}$

In [3]  $e^{At}$  is obtained as the unique solution to the following initial value problem. With  $f(\lambda)$  as specified above and  $D \equiv \frac{d}{dt}$  we wish to obtain the solution to:

$f(D)G(t) = 0$ ,  $G(t)$  an  $n \times n$  matrix with elements functions of  $t$  and such that  $G(0) = I$ ,  $G'(0) = A$ ,  $\dots$   $G^{(n-1)}(0) = A^{n-1}$ .

If  $e^{At}$  is defined by the equation

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!},$$

then

$$\left. e^{At} \right|_{t=0} = I, \left. D(e^{At}) \right|_{t=0} = A, \dots \left. D^{n-1}(e^{At}) \right|_{t=0} = A^{n-1},$$

and  $f(D)e^{At} = f(A)e^{At} = 0$  by the Cayley-Hamilton Theorem. Therefore,  $e^{At}$  is the unique solution to this initial value problem. Suppose  $\alpha_1, \alpha_2, \dots, \alpha_s$  are the distinct eigenvalues of  $A$  with multiplicities  $\mu_1, \mu_2, \dots, \mu_s$ , we may write

$$(i) \quad e^{At} = \sum_{k=1}^s (C_{k1} + C_{k2}t + \dots + C_{k\mu_k} t^{\mu_k-1}) e^{\alpha_k t},$$

where the  $C_{kj}$  are  $n \times n$  matrices. From the initial conditions we have:

$$I = \sum_{k=1}^S C_{k1}$$

$$A = \sum_{k=1}^S (\alpha_k C_{k1} + C_{k2})$$

(2)

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$A^{n-1} = \sum_{k=1}^S (\alpha_k^{n-1} C_{k1} + (n-1)\alpha_k^{n-2} C_{k2} + \dots + \frac{(n-1)!\alpha_k^{n-\mu_k}}{(n-\mu_k)!(\mu_k-1)!} C_{k\mu_k}),$$

from which the  $n$   $C_{kj}$  can be determined. We note from the system (2) that a solution for a  $C_{kj}$  will be a linear combination of the left side of (2) and therefore, any  $C_{kj}$  will be a polynomial in  $A$  of degree at most  $(n-1)$ . Being polynomials in  $A$ , the  $C_{kj}$  commute.

### 3. Properties of the $C_{k1}$ when all roots of $f(\lambda)$ are distinct.

We will not use the system (2) to completely solve for the  $n$   $C_{kj}$ , rather will we obtain another representation for  $e^{At}$  satisfying the initial value problem and then equate coefficients of like terms " $t^{\ell} e^{\alpha_k t}$ ". Before proceeding to this representation let us consider the simple case in which all roots of  $f(\lambda)$  are distinct. This case will provide insights on how to handle the more general case of multiple roots.

If all roots of  $f(\lambda)$  are distinct, then (1) becomes:

$$(1)' \quad e^{At} = \sum_{k=1}^n C_{k1} e^{\alpha_k t}.$$

The inverse of (1)' is

$$(1)'' \quad e^{-At} = \sum_{k=1}^n C_{k1} e^{-\alpha_k t}.$$

Multiplying (1)' and (1)" and using the commutative property for the  $C_{k1}$  we obtain

$$(3) \quad I - \sum_{k=1}^n C_{k1}^2 = \sum_{k=1}^n \sum_{\substack{j=1 \\ k \neq j}}^n C_{k1} C_{j1} e^{(\alpha_k - \alpha_j)t}.$$

Equation (3) is true for all  $t$ , however, the left side of (3) is independent of  $t$  which suggests that  $C_{k1} C_{j1} = 0$  for  $k \neq j$ , i.e., the  $C_{k1}$  are pairwise orthogonal. (This will be shown below.)

Applying the initial conditions to (1)' we obtain:

$$I = C_{11} + C_{21} + \dots + C_{n1}$$

$$A = \alpha_1 C_{11} + \alpha_2 C_{21} + \dots + \alpha_n C_{n1}$$

(2)'

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.

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$$A^{n-1} = \alpha_1^{n-1} C_{11} + \alpha_2^{n-1} C_{21} + \dots + \alpha_n^{n-1} C_{n1}.$$

Since the  $\alpha_k$  are distinct, the coefficient matrix for the system (2)' is a Vandermonde matrix and is non-singular. Therefore, the system (2)' has unique solutions for the  $C_{k1}$ . If we solve the system (2)' for  $C_{k1}$ , then the coefficient of  $A^{n-1}$  in this solution is:

$$(4) \quad (-1)^{n+k} \left| \begin{array}{cccc} 1 & \dots & 1 & 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_{k-1} & \alpha_{k+1} & \dots & \alpha_n \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{n-2} & \dots & \alpha_{k-1}^{n-2} & \alpha_{k+1}^{n-2} & \dots & \alpha_n^{n-2} \end{array} \right| \bigg/ \left| \begin{array}{ccc} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{array} \right|$$

Both numerator and denominator of this coefficient are non-zero Vandermonde determinants and, therefore,  $C_{k1}$  ( $k = 1, 2, \dots, n$ ) is a polynomial



in  $A$  of degree precisely  $n-1$ . Moreover, the coefficient of  $A^{n-1}$  in this solution for  $C_{k1}$  is

$$1 / \prod_{\substack{j=1 \\ j \neq k}}^n (\alpha_k - \alpha_j) ,$$

which agrees with the expansion of (4).

Next, suppose we wish to eliminate  $C_{k1}$  from the second through the  $n^{\text{th}}$  equation in (2)', which yields  $n-1$  equations in the  $n-1$  unknown matrices  $C_{11}, C_{21}, \dots, C_{k-1,1}, C_{k+1,1}, \dots, C_{n1}$ . We do this in order: subtract  $\alpha_k$  times the first equation from the second,  $\alpha_k^2$  times the first from the third, to finally  $\alpha_k^{n-1}$  times the first from the  $n^{\text{th}}$ . Deleting the first of this new system of equations we obtain  $n-1$  equations with the unknown  $C_{k1}$  missing. The left members of this new system will be  $A - \alpha_k I, A^2 - \alpha_k^2 I, \dots, A^{n-1} - \alpha_k^{n-1} I$ , each of which has a factor  $A - \alpha_k I$ , as does any linear combination of these left members. Therefore, each of the solutions for  $C_{11}, C_{21}, \dots, C_{k-1,1}, C_{k+1,1}, \dots, C_{n1}$  will have a factor  $A - \alpha_k I$ . From this we conclude further that:

$$(5) \quad C_{j1} = a_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (A - \alpha_\ell I) , \quad j = 1, 2, \dots, n$$

for some scalar  $a_j$ . We note from (5) that  $C_{j1}$  is precisely a polynomial of degree  $(n-1)$  in  $A$  with leading coefficient  $a_j$  which must be

$$(6) \quad a_j = 1 / \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\alpha_j - \alpha_\ell) \quad (\text{from (4)}) .$$

As an example using (5) and (6), let  $A$  be any  $3 \times 3$  matrix with distinct eigenvalues  $\alpha_1, \alpha_2, \alpha_3$ , then;



$$e^{At} = \frac{(A - \alpha_2 I)(A - \alpha_3 I)e^{\alpha_1 t}}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_3 I)e^{\alpha_2 t}}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_2 I)e^{\alpha_3 t}}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}.$$

### 3.1 Orthogonality and Idempotency of the $C_{k1}$ .

A further conclusion to be made at this point occurs when, using (5), we multiply  $C_{j1}$  and  $C_{i1}$  ( $i \neq j$ ); this yields

$$\begin{aligned} C_{j1}C_{i1} &= a_j a_i \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (A - \alpha_\ell I) \prod_{\substack{\ell=1 \\ \ell \neq i}}^n (A - \alpha_\ell I) \\ &= a_j a_i f(A) \prod_{\substack{\ell=1 \\ \ell \neq i, \ell \neq j}}^n (A - \alpha_\ell I) = 0 \end{aligned}$$

by the Cayley-Hamilton Theorem. Therefore, as conjectured earlier, the  $C_{k1}$  are pairwise orthogonal when the eigenvalues of  $A$  are distinct.

Using the first of the initial conditions in (2)' and this orthogonality we have  $C_{j1} = C_{j1} \sum_{k=1}^n C_{k1} = C_{j1}^2$ ,  $j = 1, 2, \dots, n$ , i.e., each  $C_{j1}$  is idempotent.

We summarize section (3) in the Theorem I. Given  $A$  and  $f(\lambda)$  with distinct eigenvalues  $\alpha_1, \alpha_2, \dots, \alpha_n$  we have

$$a) \quad e^{At} = \sum_{k=1}^n C_{k1} e^{\alpha_k t}$$

$$b) \quad C_{k1} = \prod_{\substack{\ell=1 \\ \ell \neq k}}^n \frac{(A - \alpha_\ell I)}{(\alpha_k - \alpha_\ell)} \quad k = 1, 2, \dots, n$$

$$c) \quad C_{i1}C_{j1} = \begin{cases} C_{i1} & \text{if } i = j \text{ (Idempotent)} \\ 0 & \text{if } i \neq j \text{ (Orthogonal)} \end{cases}$$

d) All  $C_{k1}$  are polynomials of degree  $n - 1$  in  $A$  and they commute.

### 4. Properties of the $C_{kj}$ when $f(\lambda)$ has multiple roots.

#### 4.1 Minimal Polynomial for a given matrix.

First consider the example:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for which } f(\lambda) = (\lambda - 1)^4.$$

According to (1)

$$e^{At} = (C_{11} + C_{12}t + C_{13}t^2 + C_{14}t^3)e^t.$$

Using the initial conditions we obtain:

$$C_{11} = I, C_{12} = (A - I), C_{13} = \frac{1}{2!} (A - I)^2, C_{14} = \frac{1}{3!} (A - I)^3.$$

However:

$$C_{13} = (A - I)^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = 0,$$

and, therefore,  $C_{13} = C_{14} = 0$ . For the given matrix  $(\lambda - 1)^4 = 0$  is the characteristic equation and  $(A - I)^4 = 0$ . But it is also true for this matrix that  $(A - I)^3 = (A - I)^2 = 0$  and  $A - I \neq 0$ . In this case  $A$  not only satisfies its characteristic equation, it also satisfies the equations  $(\lambda - 1)^3 = 0$  and  $(\lambda - 1)^2 = 0$ . For a general square matrix  $A$ , the lowest degree monic (leading coefficient equal to 1) polynomial that  $A$  satisfies is called the minimal polynomial for  $A$ . In the example above  $(\lambda - 1)^2$  is the minimal polynomial for the given  $A$  and for this matrix  $(D - 1)^2 e^{At} = 0$ . Our solution for  $e^{At}$  should then have been written

$$e^{At} = (C_{11} + C_{12}t)e^t.$$

In general, if  $\psi(\lambda)$  is the minimal polynomial for a given matrix  $A$  and the degree of  $\psi(\lambda)$  is  $m(\leq n)$ , then  $e^{At}$  satisfies the initial value problem:

$$\psi(D)e^{At} = 0$$

and

$$D^k(e^{At})_{t=0} = A^k \quad k = 0, 1, 2, \dots, m-1.$$

#### 4.2 A Redefining of the $C_{kj}$

Let the distinct eigenvalues of  $A$  be  $\alpha_1, \alpha_2, \dots, \alpha_s$  with  $\psi(\lambda) = \prod_{k=1}^s (\lambda - \alpha_k)^{\mu_k}$  the minimal polynomial for  $A$  (written in factored form),  $\mu_k \geq 1$ , and  $\sum_{k=1}^s \mu_k = m (\leq n)$  the degree of  $\psi(\lambda)$ . Following the theory presented in [2] we define  $X_k(\lambda) = \psi(\lambda)/(\lambda - \alpha_k)^{\mu_k}$ ; then  $X_k(\lambda)$  and  $(\lambda - \alpha_k)^{\mu_k}$  are relatively prime. Therefore, there exist polynomials  $p_k(\lambda), q_k(\lambda)$  (degree of  $p_k(\lambda) < \mu_k$ ) such that:

$$(7) \quad p_k(\lambda)X_k(\lambda) + q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} \equiv 1, \quad k = 1, 2, \dots, s.$$

Then define:

$$(8) \quad \begin{aligned} E_k(\lambda) &= p_k(\lambda)X_k(\lambda) \quad \text{and} \\ E_k(A) &= E_k = p_k(A)X_k(A) \quad k = 1, 2, \dots, s. \end{aligned}$$

We note from (8) that for  $k \neq \ell$   $E_k(\lambda)E_\ell(\lambda)$  is a polynomial multiple of  $\psi(\lambda)$  and, therefore,  $E_k \cdot E_\ell = 0$  for  $k \neq \ell$  (i.e. the  $E_k$  are pairwise orthogonal).

From (7) we form the product:

$$\prod_{k=1}^s q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} = \psi(\lambda) \prod_{k=1}^s q_k(\lambda) = \prod_{k=1}^s (1 - E_k(\lambda)).$$

Replacing  $\lambda$  by  $A$  in this last equation we obtain

$$\psi(A) \prod_{k=1}^s q_k(A) = 0 = \prod_{k=1}^s (I - E_k) = I - \sum_{k=1}^s E_k,$$

which follows from the definition of  $\psi(\lambda)$  and the orthogonality of the  $E_k$ . From this we have

$$(9) \quad I = \sum_{k=1}^s E_k.$$

Multiplying through (9) by  $E_\ell$ , using the orthogonality, we have

$$E_\ell = E_\ell^2 \quad \ell = 1, 2, \dots, s$$

i.e., the  $E_\ell$  are idempotent.

Next, define:

$$(10) \quad \begin{aligned} N_k(\lambda) &= (\lambda - \alpha_k)E_k \quad \text{and} \\ N_k(A) &= N_k = (A - \alpha_k I)E_k \quad (\text{for } \mu_k > 1). \end{aligned}$$

If  $\mu_k = 1$ , then  $\alpha_k$  is a simple root of  $\psi(\lambda)$  and for such roots

$$N_k(A) = \psi(A) = 0. \quad \text{We note from (10) that } N_k^{\mu_k} = (A - \alpha_k I)^{\mu_k} E_k = \psi(A) = 0$$

and  $N_k^{\mu_k-1} \neq 0$ . The  $N_k$  are said to be nilpotent of index  $\mu_k$ .

Additional conclusions from definitions (8) and (10) are:

(a) All  $E_k, N_k$  are polynomials in  $A$  and, therefore, commute.

(b)  $E_k N_k = N_k, E_k N_j = N_k N_j = 0$  ( $k \neq j$ ).

(c) We have the identity:

$$(11) \quad A = \sum_{k=1}^s E_k (\alpha_k I + N_k)$$

which can be seen as follows:

$$\begin{aligned} \sum_{k=1}^s E_k (\alpha_k I + N_k) &= \sum_{k=1}^s (\alpha_k E_k + A E_k - \alpha_k E_k) \\ &= \sum_{k=1}^s A E_k = A \sum_{k=1}^s E_k = A I = A. \end{aligned}$$

If we replace  $A$  in  $e^{At}$  by the identity (11), we have:

$$(12) \quad \begin{aligned} e^{At} &= e^{\left( \sum_{k=1}^s E_k (\alpha_k I + N_k) \right) t} = e^{\sum_{k=1}^s E_k \alpha_k t} \cdot e^{\sum_{k=1}^s N_k t} \\ &= \sum_{k=1}^s e^{\alpha_k t} \left\{ E_k + N_k t + \frac{(N_k t)^2}{2!} + \dots + \frac{(N_k t)^{\mu_k-1}}{(\mu_k-1)!} \right\}, \end{aligned}$$

and this must be identical with (1) i.e.

$$= \sum_{k=1}^s e^{\alpha_k t} \{ C_{k1} + C_{k2} t + \dots + C_{k\mu_k} t^{\mu_k-1} \}$$

Rewriting (12) using the definition (10) we have:

$$(13) \quad e^{At} = \sum_{k=1}^s E_k e^{\alpha_k t} \left\{ I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1}}{(\mu_k - 1)!} t^{\mu_k - 1} \right\}.$$

In (13) we observe that we have only the  $E_k$  ( $k = 1, 2, \dots, s$ ) to determine. These can be calculated from the definition (8) or from (2) calculating only the  $C_{k1} = E_k$  ( $k = 1, 2, \dots, s$ ).

#### 4.3 Summary of section 4 and examples.

We summarize this section in:

Theorem 2. Given  $A$  with minimal polynomial  $\psi(\lambda) = \prod_{k=1}^s (\lambda - \alpha_k)^{\mu_k}$ ,  $m \leq n$ , we have:

$$e^{At} = \sum_{k=1}^s E_k e^{\alpha_k t} \left\{ I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1}}{(\mu_k - 1)!} t^{\mu_k - 1} \right\},$$

in which the  $E_k$  satisfy the following:

$$(a) \quad I = \sum_{k=1}^s E_k$$

$$(b) \quad E_k E_j = \begin{cases} 0 & \text{if } i \neq j \\ E_k & \text{if } k = j \end{cases}$$

$$(c) \quad (A - \alpha_k I)E_k = N_k \text{ are nilpotent of index } \mu_k$$

$$(d) \quad \text{all } E_k \text{ and } N_k \text{ are polynomials in } A.$$

Up to this point we have determined  $e^{At}$  for any  $A(3 \times 3)$  with distinct eigenvalues. Now let us complete this calculation for any  $3 \times 3$  matrix  $A$ . To this end we have the following cases and calculations:

$$(i) \quad \text{All eigenvalues of } A \text{ are equal to } \alpha, \text{ and } \psi(\lambda) = f(\lambda) = (\lambda - \alpha)^3.$$

In this case:

$$e^{At} = e^{\alpha t} \left( E_1 + N_1 t + \frac{N_1^2 t^2}{2!} \right) \text{ in which } E_1 = I \text{ and } N_1 = (A - \alpha I).$$

(ii) Again all eigenvalues equal  $\alpha$ , but  $\psi(\lambda) = (\lambda - \alpha)^2$ . In this case  $E_1 = I$  and  $e^{At} = e^{\alpha t}(I + (A - \alpha I)t)$ .

(iii)  $\psi(\lambda) = (\lambda - \alpha)$ . In this case  $E_1 = I$  and  $e^{At} = e^{\alpha t}I$  a scalar matrix.

(iv)  $\psi(\lambda) = f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$ ,  $\alpha_1 \neq \alpha_2$ .

In this case:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2.$$

We can solve for  $E_1$  and  $E_2$  ( $N_1 = (A - \alpha_1 I)E_1$ ) using the initial conditions (2) or definitions (7) and (8). In view of (7) we have (replacing  $\lambda$  by  $A$ ):

$$E_1 + q_1(A)(A - \alpha_1 I)^2 = I.$$

However, we know that  $E_1 + E_2 = I$  and, therefore,  $E_2 = q_1(A)(A - \alpha_1 I)^2$  which means we obtain both  $E_1$  and  $E_2$  simultaneously by using (7) and (8). Accordingly using (7) and (8):  $(a\lambda + b)(\lambda - \alpha_2) + c(\lambda - \alpha_1)^2 \equiv 1$  which must hold for all  $\lambda$  and, therefore, we have:

$$a = \frac{-1}{(\alpha_2 - \alpha_1)^2}, \quad b = \frac{-(\alpha_2 - 2\alpha_1)}{(\alpha_2 - \alpha_1)^2}, \quad c = \frac{1}{(\alpha_2 - \alpha_1)^2}.$$

Therefore:

$$E_1 = \frac{-1}{(\alpha_2 - \alpha_1)^2} (A - \alpha_2 I)(A + (\alpha_2 - 2\alpha_1)I)$$

$$E_2 = \frac{1}{(\alpha_2 - \alpha_1)^2} (A - \alpha_1 I)^2 \quad \text{and}$$

$$N_1 = \frac{-(A - \alpha_1 I)}{(\alpha_2 - \alpha_1)^2} (A - \alpha_2 I)(A - 2\alpha_1 I + \alpha_2 I).$$

Accordingly:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2.$$

(v) The last case is that in which  $f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$  as in (iv) but  $\psi(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)$ .

In this case  $\alpha_1$  and  $\alpha_2$  are simple roots of  $\psi(\lambda)$  and therefore,

$$e^{At} = \frac{e^{\alpha_1 t}}{\alpha_1 - \alpha_2} (A - \alpha_2 I) + \frac{e^{\alpha_2 t}}{\alpha_2 - \alpha_1} (A - \alpha_1 I)$$

by virtue of the results in section 3.

In view of the example in section 3 and the 5 cases above we have obtained  $e^{At}$  for any 3 x 3 matrix.

It is of interest to note that for any  $A(n \times n)$  in which either (a)  $A$  has  $n$  equal eigenvalues or the opposite extreme (b)  $A$  has  $n$  distinct eigenvalues we have that  $e^{At}$  can be written immediately as:

$$(a) e^{At} = e^{\alpha t} \sum_{k=0}^{n-1} \frac{(A - \alpha I)^k t^k}{k!}.$$

If in this case  $\psi(\lambda) = (\lambda - \alpha)^m$ ,  $m < n$ , then the summation would extend only to  $m - 1$  since  $(A - \alpha I)^m = 0$ .

$$(b) e^{At} = \sum_{k=1}^n e^{\alpha_k t} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{(A - \alpha_j I)}{(\alpha_k - \alpha_j)}.$$

##### 5. Other representations for $e^{At}$ .

In [4] and [6]  $e^{At}$  is obtained by use of the Lagrange-Sylvester interpolation polynomial. By using the eigenvalues of  $A$  as the interpolation points we obtain the form of  $e^{At}$  in equation (12). In [5] and [7] representations of  $e^{At}$  are obtained in one case in powers of  $A$  and in others in powers of  $A - \alpha_i I$  ( $\alpha_i$  - eigenvalues of  $A$ ). In any case, if all these representations were given in powers of the same  $(A - \beta_i I)$  then they would, of course, all be the same.

Another representation for  $e^{At}$ , which has applications to solutions of first order linear systems of simultaneous differential equations



with constant coefficients, is obtained as follows. Suppose we have given:

$$(14) \quad x'(t) = Ax(t)$$

A an  $n \times n$  matrix with constant elements,  $x(t)$  an  $n \times 1$  vector function of  $t$ . Suppose we have found a fundamental set of solutions for (14) namely  $x_1(t), x_2(t), \dots, x_n(t)$ . Then define:

$$(15) \quad X(t) = (x_1(t), x_2(t), \dots, x_n(t)) ,$$

which is an  $n \times n$  matrix whose columns are the elements of the fundamental set.  $X(0)$  is nonsingular and we define

$$(16) \quad G(t) = X(t)(X(0))^{-1};$$

then

$$G(t) = e^{At}.$$

This equation follows from differentiating (16), which yields

$$G'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))(X(0))^{-1} .$$

By (14)  $G'(t)$  can be written:

$$\begin{aligned} G'(t) &= (Ax_1(t), Ax_2(t), \dots, Ax_n(t))(X(0))^{-1} \\ &= AX(t)(X(0))^{-1} = AG(t) . \end{aligned}$$

From this last equation it follows that:

$$G^{(k)}(t) = A^k G(t) \quad k=0, 1, \dots,$$

which in turn yields:

$$G^{(k)}(0) = A^k \quad k=0, 1, \dots .$$

Moreover,

$$f(D)G(t) = f(A)G(t) = 0$$

by the Cayley-Hamilton theorem.

Therefore,  $G(t)$  satisfies the initial value problem which is also satisfied by  $e^{At}$ . By uniqueness of such solutions we conclude:

$$G(t) = X(t)(X(0))^{-1} = e^{At}.$$

## 6. Some applications of $e^{At}$ .

From the last remarks in section 5, and since  $(X(0))^{-1}$  is nonsingular,  $X(t)(X(0))^{-1} = e^{At}$  has columns that are linear combinations of the columns of  $X(t)$  and form another fundamental set for the differential equation (14):

$$X'(t) = AX(t),$$

Let us use this fact to solve the system

$$X'(t) = \begin{pmatrix} 4 & -5 & 3 \\ 2 & -3 & 2 \\ -1 & 1 & 0 \end{pmatrix} X(t),$$

The given matrix has characteristic polynomial  $f(\lambda) = \psi(\lambda) = (\lambda - 1)^2(\lambda + 1)$

Using case (iv) in section 4 with  $\alpha_1 = 1$  and  $\alpha_2 = -1$  we have:

$$\begin{aligned} e^{At} &= e^t \left\{ \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} t \right\} + e^{-t} \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t(2+t) - e^{-t} & -e^t(2+t) + 2e^{-t} & e^t(1+t) - e^{-t} \\ e^t - e^{-t} & -e^t + 2e^{-t} & e^t - e^{-t} \\ -te^t & te^t & e^t(1-t) \end{pmatrix} \end{aligned}$$

The columns of the latter matrix constitute a fundamental set for the differential equation and its general solution is:

$$X(t) = e^{At} X(0),$$

where  $X(0)$  forms the initial conditions for  $X(t)$  (given at  $t = 0$ ).

As a matter of fact, having obtained  $e^{At}$  for all  $3 \times 3$  matrices  $A$  we have therefore obtained fundamental sets for all systems of differential equations  $X'(t) = AX(t)$  with  $A$  a  $3 \times 3$  matrix of constants and  $X(t)$  a  $3 \times 1$  vector function of  $t$ .

Knowing  $e^{At}$  and  $e^{-At}$  we define

$$\cosh At = 1/2(e^{At} + e^{-At}) \quad \text{and}$$

$$\sinh At = 1/2(e^{At} - e^{-At}).$$

Equally well we know  $e^{iAt}$  and  $e^{-iAt}$  ( $i = \sqrt{-1}$ ) and define:

$$\cos At = 1/2(e^{iAt} + e^{-iAt})$$

$$\sin At = \frac{1}{2i}(e^{iAt} - e^{-iAt}).$$

As an example, we indicate the expansion of  $\cosh At$ :

$$\cosh At = \sum_{k=1}^s \cosh \alpha_k t (E_k + \frac{N_k^2 t^2}{2!} + \dots) + \sum_{k=1}^s \sinh \alpha_k t (N_k t + \frac{N_k^3 t^3}{3!} + \dots)$$

with each of these two sums terminating with the term containing either  $t^{\mu_k-2}$  or  $t^{\mu_k-1}$ , depending upon whether  $\mu_k$  is even or odd.

In [1] T. M. Apostol considers the system of differential equations

$$Y''(t) = AY(t)$$

and writes the solution in terms of two matrix functions

$$C(t) = \sum_{k=0}^{\infty} \frac{t^{2k} A^k}{(2k)!}, \quad S(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1} A^k}{(2k+1)!}.$$

$C(t)$  is precisely  $\cosh \sqrt{A} t$  and  $S(t)$  is  $(\sqrt{A})^{-1} \sinh \sqrt{A} t$  provided that  $\sqrt{A}$  is defined and nonsingular. Clearly one would define  $\sqrt{A}$  to be that matrix  $B$  such that  $B^2 = A$ . It turns out that  $B$  is not unique (as one would suspect), in fact, there may be as many as  $2^n$  matrices  $B$  such that  $B^2 = A$ . However, these  $B$ -matrices may be calculated as follows. If  $A$  is similar to a diagonal matrix so that

$$A = T^{-1} \text{Diag} \{\alpha_1, \alpha_2, \dots, \alpha_n\} T,$$

then

$$\sqrt{A} = T^{-1} \text{Diag} \{\pm \alpha_1^{1/2}, \pm \alpha_2^{1/2}, \dots, \pm \alpha_n^{1/2}\} T.$$

If  $A$  is not similar to a diagonal matrix and is nonsingular, then  $\sqrt{A}$  can be obtained as follows: From (11)

$$A = \sum_{k=1}^s E_k (\alpha_k I + N_k) = \sum_{k=1}^s E_k \alpha_k (I + \frac{N_k}{\alpha_k}) ;$$

then

$$\sqrt{A} = \sum_{k=1}^s \pm \alpha_k^{1/2} E_k (I + \frac{N_k}{\alpha_k})^{1/2}$$

If  $(I + \frac{N_k}{\alpha_k})^{1/2}$  is expanded by the binomial theorem then this expansion would terminate with the term  $N_k^{\mu_k-1}$  since  $N_k$  is nilpotent of index  $\mu_k$ .

Knowing how to compute  $\sqrt{A}$  in some cases we then have for these cases the solutions to

$$(a) \quad Y''(t) + AY(t) = 0$$

given by

$$Y(t) = (\cos \sqrt{A} t) Y_1 + (\sin \sqrt{A} t) Y_2;$$

or

$$(b) \quad Y''(t) - AY(t) = 0$$

given by

$$Y(t) = (\cosh \sqrt{A} t) Y_1 + (\sinh \sqrt{A} t) Y_2.$$

In (a) and (b)

$$Y(0) = Y_1 \text{ and } Y'(0) = \sqrt{A} Y_2.$$

As a last application we will consider: let  $A$  be a  $3 \times 3$  matrix with characteristic function  $f(\lambda) = \psi(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$  (case (iv) in section 4), and suppose we are given the nonhomogeneous system

$$(17) \quad x'(t) = Ax(t) + a_1 e^{\alpha_1 t},$$

in which  $x(t)$  is a  $3 \times 1$  vector function of  $t$  to be determined and  $a_1$  is a  $3 \times 1$  constant vector. Multiplying through (17) by  $e^{-At}$  yields

$$e^{-At}x'(t) - Ae^{-At}x(t) = D(e^{-At}x(t)) = e^{-At}a_1e^{\alpha_1 t}.$$

Integrating this last equation we obtain

$$(18) \quad x(t) = e^{At} \int^t e^{-A\tau} a_1 e^{\alpha_1 \tau} d\tau + e^{At} a_2,$$

in which  $a_2$  is a  $3 \times 1$  vector whose elements are arbitrary constants.

From case (iv)

$$(19) \quad e^{At} = e^{\alpha_1 t} (E_1 + N_1 t) + e^{\alpha_2 t} E_2.$$

Substituting (19) into (18), using the orthogonality, idempotency and nilpotency of  $E_1$ ,  $E_2$  and  $N_1$  and integrating we obtain

$$x(t) = e^{\alpha_1 t} \left\{ \left( E_1 + \frac{N_1 t}{2} \right) t + \frac{E_2}{\alpha_1 - \alpha_2} \right\} a_1 + \{ e^{\alpha_1 t} (E_1 + N_1 t) + e^{\alpha_2 t} E_2 \} a_2$$

as the complete solution to (17).

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